

INFINITARY BRAID GROUPS

KATSUYA EDA AND TAKESHI KANETO

This preprint was written before 1993 when the first author was in University of Tsukuba. Now, according to the new result by W. Herfort and W. Hojka the conclusion of Lemma 2.11 and hence that of Theorem 2.8 becomes “cotorsion” instead of “complete mod-U”.

1. INTRODUCTION AND DEFINITIONS

An infinitary version of braid groups has been implicitly considered as a direct limit of n -braid groups (see Remark 1.5). However, as an intuitive object, we can imagine a more complicated braid with infinitely many strings (see Figure (*)).

In this paper we introduce an infinitary version of braid groups and study about fundamental properties of it especially in case the number of strings is countable. Our notation and notion are usual ones for braid groups and topology [2, 8, 11]. \mathbb{Z}, \mathbb{R} and \mathbb{C} are the set of integers, real numbers and complex numbers, respectively. The unit interval \mathbb{I} is $[0, 1]$. For a path f in a space X , i.e. a continuous map $f : \mathbb{I} \rightarrow X$, f^- is the path defined by $f^-(t) = f(1 - t)$. For paths $f, g : \mathbb{I} \rightarrow X$ with $f(1) = g(0)$, $f \cdot g$ is the path defined by: $f \cdot g(t) = f(2t)$ for $0 \leq t \leq 1/2$ and $f \cdot g(t) = g(2t - 1)$ for $1/2 \leq t \leq 1$. For paths f and g , $f \sim g$ if f and g are homotopic relative to $\{0, 1\}$. A constant path is denoted by the point as usual.

For a subset X of \mathbb{C} let \mathbb{C}^X be the product space, regarding X as an indexed set. For distinct $x, y \in X$, let $\Delta_{xy} = \{u \in \mathbb{C}^X : u(x) = u(y)\}$ and $\Delta_X = \bigcup \{\Delta_{xy} : x \neq y, x, y \in X\}$. The subspace $\mathbb{C}^X \setminus \Delta_X$ of \mathbb{C}^X is denoted by F_X . For a path f in F_X and $Y \subset X \subset \mathbb{C}$, the restriction $f|_Y$ is a path in F_Y , where $f|_Y(t)(y) = f(t)(y)$ for $y \in Y$. Let Σ_X be the group consisting of all permutations on the discrete set X , which also acts freely on F_X as coordinate transformations. For a path f in F_X and $\sigma \in \Sigma_X$, f^σ is the path in F_X defined by: $f^\sigma(t)(x) = f(t)(\sigma(x))$ for $t \in \mathbb{I}, x \in X$.

Definition 1.1. For a subset X of \mathbb{C} , an X -braid $f : \mathbb{I} \rightarrow F_X$ is a path satisfying $f(0) = \text{id}_X$ and $f(1) \in \Sigma_X$. In case $f(1) = \text{id}_X$, f is called a pure X -braid. X -braids f and g are equivalent, if $f \sim g$. Let $[f]$ be the equivalence class containing f with respect to \sim for an X -braid f . Let A_X be the set of X -braids, $B_X = \{[f] : f \in A_X\}$ and $P_X = \{[f] : f \text{ is a pure } X\text{-braid}\}$. For $f, g \in A_X$, let $f \# g = f \cdot g^{f(1)}$.

In case X is finite, the X -braid group is defined as the fundamental group of the orbit space F_X/Σ_X in [6]. Then we can rewrite this definition in terms of paths in

The authors thank J. Morita, A. Tsuboi and K. Sakai for stimulating talks.

F_X , for F_X is a covering space of F_X/Σ_X (see Remark 1.5 (2)). Analogously, we define the X -braid group B_X for a finite or infinite set X in the sense of the next proposition, where the subgroup P_X of B_X is just the fundamental group of a space F_X .

Proposition 1.2. *Let $[f][g] = [f\#g]$ for $f, g \in A_X$. Then, this operation is well-defined and B_X becomes a group.*

Proof. Since the operation is clearly well-defined, we show the existence of identity and inverse. The equivalence class of the constant map $[\text{id}_X]$ is clearly the left identity. For $f \in A_X$, let $\sigma = f(1)^{-1} \in \Sigma_X$. Then, $(f^-)^\sigma \in A_X$ and $[f][(f^-)^\sigma] = [f \cdot (f^-)^{\sigma f(1)}] = [f \cdot f^-] = e$ and hence $[(f^-)^\sigma]$ is the left inverse of $[f]$. \square

To simplify the notation, we adopt some convention in set theory, i.e. a non-negative integer n is the set $\{0, \dots, n-1\}$, 0 is the empty set and ω is the set of all non-negative integers. Hence, $m < n$ if and only if $m \in n$ for $m, n \in \omega$. We denote the set of positive integers by ω_+ . Then, an n -braid in the usual sense is the same as defined as above and we can see that the X -braid group B_X coincides with the usual n -braid group if $X = n$. We identify $f \in A_X$ with the indexed family of paths $(p_x : x \in X)$ in \mathbb{C} satisfying $p_x(0) = x$ for $x \in X$, $p_x(t) \neq p_y(t)$ for $x \neq y \in X$ and $\{p_x(1) : x \in X\} = X$, i.e. $p_x(t) = f(t)(x)$. Then, the restriction $f|_Y$ for $Y \subset X$ is the subfamily $(p_x : x \in Y)$ of $f = (p_x : x \in X)$. we call p_x the x -string. Since a homotopy in a product space corresponds to homotopies in all components, we get the next proposition, which is obtained just by rewriting the definition of the equivalence \sim .

Proposition 1.3. *For X -braids $f = (p_x : x \in X), g = (q_x : x \in X)$, f and g are equivalent if and only if there exists an indexed family of homotopies $(H_x : x \in X)$ such that*

- (1) $H_x : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{C}$ is continuous;
- (2) $H_x(s, t) \neq H_y(s, t)$ for $s, t \in \mathbb{I}, x \neq y$;
- (3) $H_x(0, t) = x, H_x(1, t) = p_x(1) = q_x(1)$ for $t \in \mathbb{I}$;
- (4) $H_x(s, 0) = p_x(s)$ and $H_x(s, 1) = q_x(s)$ for $s \in \mathbb{I}$.

Now, we can see that X -braids are “strongly isotopic” in the sense of Artin [1] if and only if they are equivalent, in case X is finite.

Definition 1.4. Two X -braids f and g are strongly equivalent, if there exists a continuous map $H : \mathbb{C} \times \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{C}$ such that

- (1) $H_{s,t} : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism, where $H_{s,t}(\alpha) = H(\alpha, s, t)$;
- (2) $H(\alpha, s, 0) = H(\alpha, 0, t) = H(\alpha, 1, t) = \alpha$ for $\alpha \in \mathbb{C}, s, t \in \mathbb{I}$;
- (3) $H(f(s)(x), s, 1) = g(s)(x)$ for $s \in \mathbb{I}, x \in X$.

In case X is finite, E. Artin [1, Theorem 6] showed that X -braids f, g are equivalent if and only if they are strongly equivalent. More precisely, he showed that

for given H_x ($x \in X$) in Proposition 1.3 there exists H in Definition 1.4 such that $H(f(s)(x), s, t) = H_x(s, t)$ and $H(\alpha, s, t) = \alpha$ if $|\alpha|$ is large enough.

Remark 1.5. (1) If we induce the box product topology to the space \mathbb{C}^ω , the situation changes as follows. The path connected component of id_ω consists of all $u \in F_\omega$ such that $u(n) = n$ for all but finite $n \in \omega$. Therefore, the group obtained by the procedure in Definition 1.1 and Proposition 1.2 in this case is the sum $\bigcup \{B_n : n \in \omega\}$ under the natural identification of B_n as a subgroup of B_{n+1} and the homomorphic image of the canonical map to Σ_ω consists of permutations of finite support.

(2) One might suspect why we do not treat with the quotient space of F_X using Σ_X as in [6]. If we take such a quotient of A_X in Definition 1.1 in case that X is infinite, the quotient space does not become Hausdorff nor the quotient map is a regular cover. These are the reasons.

2. BASIC RESULTS OF B_X AND ω -BRAIDS

To show some results of B_X the next proposition is necessary. For its proof we debt A. Tsuboi.

Proposition 2.1. *For any $X \subset \mathbb{C}$, the space F_X is path connected.*

Proof. Let $u \in F_X$. We shall choose $(\alpha, 1/2) \in \mathbb{I} \times \mathbb{I}$ for each $x \in X$ so that the dogleg segments from $(x, 0)$ to $(u(x), 1)$ via $(\alpha, 1/2)$ do not intersect for distinct x 's. First, well-order X so that $X = \{x_\mu : \mu < \kappa\}$ and the cardinality of $\{\nu : \nu < \mu\}$ is less than 2^{\aleph_0} for each $\mu < \kappa$. Let $Q_\mu = (x_\mu, 0)$ and $S_\mu = (u(x_\mu), 1)$ for $\mu < \kappa$. Suppose that we have gotten $R_\nu \in \mathbb{C} \times \{1/2\}$ ($\nu < \mu$) so that $Q_\nu R_\nu S_\nu$ do not intersect mutually for distinct ν 's. Since there are 2^{\aleph_0} -many planes which contain Q_μ and S_μ , there exists a plane which contains Q_μ and S_μ but does neither contain any Q_ν nor S_ν ($\nu < \mu$). Take such a plane. Then, the intersection of the plane and $Q_\nu R_\nu S_\nu$ consists of at most two points for each $\nu < \mu$. Therefore, there are less than 2^{\aleph_0} -many points on the plane which are on some constructed $Q_\nu R_\nu S_\nu$. Tracing $(\alpha, 1/2)$'s in the plane, we get 2^{\aleph_0} -many distinct dogleg segments in the plane which connect x_μ and $u(x_\mu)$. Therefore, we can choose R_μ so that $Q_\mu R_\mu S_\mu$ does not intersect with $Q_\nu R_\nu S_\nu$ for $\nu < \mu$. For $t \in \mathbb{I}$, define $f(t)(x_\mu)$ so that $(f(t)(x_\mu), t)$ is on $Q_\mu R_\mu S_\mu$. Then, $f : \mathbb{I} \rightarrow F_X$ is a path from id_X to u . \square

Theorem 2.2. *If X and Y have the same cardinality for $X, Y \subset \mathbb{C}$, B_X and B_Y are isomorphic.*

Proof. Take a bijective function $\varphi : X \rightarrow Y$. Then, φ induces a homeomorphism between F_X and F_Y . Let $\sigma_f = f(1)$ for $f \in A_X \cup A_Y$. For $f \in A_X$ and $g \in A_Y$, define paths $\varphi(f)$ in F_Y and $\varphi^{-1}(g)$ in F_X by: $\varphi(f)(t)(y) = f(t)(\varphi^{-1}(y))$ and $\varphi^{-1}(g)(t)(x) = g(t)(\varphi(x))$ for $t \in \mathbb{I}, x \in X, y \in Y$. Then $\varphi(f)$ is a path from φ^{-1} to $\sigma_f \varphi^{-1}$. By Proposition 2.1, there exists a path h from id_Y to $\varphi^{-1} \in F_Y$. Define

$\Phi : A_X \rightarrow A_Y$ by: $\Phi(f) = h \cdot \varphi(f) \cdot (h^-)^{\varphi\sigma_f\varphi^{-1}}$. It is easy to see $\Phi(f) \sim \Phi(g)$ for $f \sim g$. On the other hand, $\varphi^{-1}(h \cdot \varphi(f) \cdot (h^-)^{\varphi\sigma_f\varphi^{-1}}) = \varphi^{-1}(h) \cdot f \cdot \varphi^{-1}((h^-)^{\varphi\sigma_f\varphi^{-1}}) = \varphi^{-1}(h) \cdot f \cdot (\varphi^{-1}(h^-))^{\sigma_f}$. Therefore, $\varphi^{-1}(h^-) \cdot \varphi^{-1}(\Phi(f)) \cdot (\varphi^{-1}(h))^{\sigma_f} \in A_X$ is equivalent to f , which shows that $\Phi(f) \sim \Phi(g)$ implies $f \sim g$. Therefore, Φ induces an injection from B_X to B_Y . The surjectivity of the induced map can be proved similarly as above. The remaining thing to show is $\Phi(f\#g) \sim \Phi(f)\#\Phi(g)$.

$$\begin{aligned} \Phi(f)\#\Phi(g) &= h \cdot \varphi(f) \cdot (h^-)^{\varphi\sigma_f\varphi^{-1}} \cdot h^{\varphi\sigma_f\varphi^{-1}} \cdot \varphi(g)^{\varphi\sigma_g\varphi^{-1}} \cdot (h^-)^{\varphi\sigma_g\varphi^{-1}} \cdot \varphi\sigma_f\varphi^{-1} \\ &\sim h \cdot \varphi(f) \cdot \varphi(g)^{\varphi\sigma_f\varphi^{-1}} \cdot (h^-)^{\varphi\sigma_g\varphi^{-1}} \\ &= h \cdot \varphi(f\#g) \cdot (h^-)^{\varphi\sigma_{f\#g}\varphi^{-1}} \\ &= h \cdot \varphi(f\#g) \cdot (h^-)^{\varphi\sigma_{f\#g}\varphi^{-1}} \\ &= \Phi(f\#g). \end{aligned}$$

□

Next, we prove an exact sequence which shows a relationship between braid groups and permutation groups.

Proposition 2.3. *The following exact sequence holds for any subset X of \mathbb{C} :*

$$0 \rightarrow P_X \rightarrow B_X \rightarrow \Sigma_X \rightarrow 0.$$

Proof. Let $h : B_X \rightarrow \Sigma_X$ be the canonical homomorphism, i.e. $h([f])(x) = f(1)(x)$ for $f \in A_X, x \in X$. Then, $\text{Ker}(h) = P_X$ clearly. For $\sigma \in \Sigma_X$ there exists a path f in F_X from id_X to σ by Proposition 2.1. Then, $f \in A_X$ and $h([f]) = \sigma$. □

It is well-known that B_n are torsion-free for $n \in \omega$ [6, 3]. We shall show that B_ω is torsion-free. It follows from the next theorem, which shows a fundamental property of ω -braids.

Theorem 2.4. *Let f, g be ω -braids. Then, $f \sim g$ if and only if $f|_n \sim g|_n$ for the restrictions $f|_n, g|_n$ ($n \in \omega$).*

Corollary 2.5. *The ω -braid group B_ω is torsion-free.*

Proof. Let $[f]^m = e$ for $f \in A_\omega, m \neq 0$ and $\sigma = f(1)$. Then, $\sigma^m = e$ in Σ_ω and hence there exists a partition $\{E_n : n \in \omega\}$ of ω such that the cardinalities of E_n are divisors of m and each restriction of σ to E_n is a cyclic permutation on E_n . The restriction $f|_{E_n}$ is an E_n -braid and $[f|_{E_n}]^m = e$. Therefore, $[f|_{E_n}] = e$ and hence $\sigma|_{E_n} = e$ for each n . Now, we have shown that f is a pure ω -braid.

Since $[f|_n]^m = e$ by the assumption, $f|_n \sim \text{id}_n$ for every $n \in \omega$. Hence, $[f] = e$ by Theorem 2.4. □

To show Theorem 2.4, some lemmas are necessary.

Lemma 2.6. *Let $f \in A_n$ be a pure n -braid for $n \in \omega$ such that $[f] = e$ and $f(s)(i) = i$ for $i \in k \leq n, s \in \mathbb{I}$. Then, there exists a continuous map $H : \mathbb{C} \times \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{C}$ such*

that

- (1) $H_{s,t} : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism, where $H_{s,t}(\alpha) = H(\alpha, s, t)$;
- (2) $H(\alpha, s, 0) = \alpha$ and $H(j, s, 1) = f(s)(j)$ for $j \in n$;
- (3) $H(\alpha, 0, t) = H(\alpha, 1, t) = \alpha$;
- (4) $H(i, s, t) = i$ for $i \in k$.

Proof. It suffices to show the case $k = n-1$, since the conclusion can be obtained by repeated use of such a special case. By the assumption and [1, Theorem 6], id_n and f are strongly equivalent, that is, there exists a continuous map $H' : \mathbb{C} \times \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{C}$ satisfying:

- (1) $H'_{s,t} : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism ;
- (2) $H'(\alpha, s, 0) = \alpha$ and $H'(j, s, 1) = f(s)(j)$ for $j \in n$;
- (3) $H'(\alpha, 0, t) = H'(\alpha, 1, t) = \alpha$;
- (4) $H'(m, s, t) = m$ for large enough m .

For $\epsilon > 0$, define $\psi_\epsilon : \mathbb{I} \rightarrow [n-1, m] (\subset \mathbb{C})$ as follows:

$$\psi_\epsilon(s) = \begin{cases} (1 - \frac{s}{\epsilon})(n-1) + \frac{s}{\epsilon}m, & \text{for } 0 \leq s \leq \epsilon; \\ m, & \text{for } \epsilon < s \leq 1 - \epsilon; \\ \frac{s-1+\epsilon}{\epsilon}(n-1) + (1 - \frac{s-1+\epsilon}{\epsilon})m, & \text{for } 1 - \epsilon < s \leq 1 \end{cases}$$

Let $g_\epsilon(s) = H'(\psi_\epsilon(s), s, 1)$. Since $H'(\alpha, 0, 1) = H'(\alpha, 1, 1) = \alpha$ for any $n-1 \leq \alpha \leq m$ and $H'(m, s, 1) = m$ for any $s \in \mathbb{I}$, there exists $\epsilon > 0$ such that the real part of $g_\epsilon(s)$ is greater than $n-1-1/2$. Fix such an ϵ . Using $H'((1-t)(n-1) + t\psi_\epsilon(s), s, 1)$, we get a homotopy from f to an n -braid whose $(n-1)$ -string g_ϵ varies in $\{\alpha \in \mathbb{C} : \text{Re}(\alpha) > n-1-1/2\}$ keeping the i -strings ($i \in n-1$) fixed. Then, we can easily make the $(n-1)$ -string straight leaving the i -strings ($i \in n-1$) fixed. To perform the works altogether, let $H''_i(s, t) = i$ for $i \in n-1$ and

$$H''_{n-1}(s, t) = \begin{cases} H'((1-2t)(n-1) + 2t\psi_\epsilon(s), s, 1) & \text{for } 0 \leq t \leq 1/2; \\ (2-2t)g_\epsilon + (2t-1)(n-1) & \text{for } 1/2 < t \leq 1. \end{cases}$$

To get the desired H , let $H_i(s, t) = H''_i(s, 1-t)$. Then, H_i ($i \in n$) satisfy the following:

- (1) $H_i : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{C}$ is continuous and $H_i(s, t) \neq H_j(s, t)$ for $i \neq j$;
- (2) $H_i(s, 0) = i$ and $H_i(s, 1) = f(s)(i)$ for $i \in n$;
- (3) $H_i(0, t) = H_i(1, t) = i$ for $i \in n$;
- (4) $H_i(s, t) = i$ for $i \in n-1$.

Again by [1, Theorem 6], we can extend H_i 's to the desired H . □

Lemma 2.7. *Let f be a pure ω -braid. Then, $f \sim \text{id}_\omega$ if and only if $f|_n \sim \text{id}_n$ for every $n \in \omega$.*

Proof. It suffices to show the one direction. Suppose that $f|_n \sim \text{id}_n$ for every $n \in \omega$. By induction we define pure ω -braids f_m and a continuous map $H^m : \mathbb{C} \times \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{C}$ as the following:

- (1) $f_0 = f, f_m(s)(n) = H^m(n, s, 0), f_{m+1}(s)(n) = H^m(n, s, 1);$
- (2) $H_{s,t}^m$ is an autohomeomorphism on \mathbb{C} ,
where $H_{s,t}^m(\alpha) = H^m(\alpha, s, t);$
- (3) $H^m(\alpha, 0, t) = H^m(\alpha, 1, t) = \alpha;$
- (4) $H^m(k, s, t) = k$ for $k \in m, H^m(k, s, 1) = k$ for $k \in m+1.$

Using Lemma 2.7 and the facts $[f|_m] = e$ ($m \in \omega$) and $f_m(s)(k) = k$ ($k \in m$), we can define these above. Define $H : \mathbb{I} \times \mathbb{I} \rightarrow F_\omega$ as follows:

$$\begin{cases} H(s, 1)(n) &= n; \\ H(s, t)(n) &= H^m(n, s, 2^{m+1}(t - \sum_{i=1}^m 1/2^i)), \\ &\text{if } \sum_{i=1}^m 1/2^i \leq t \leq \sum_{i=1}^{m+1} 1/2^i. \end{cases}$$

Then, $H(0, t) = H(1, t) = \text{id}_\omega$, $H(s, 0) = f(s)$ and $H(s, 1) = \text{id}_\omega$ hold. The continuity of H follows from the fact that $H^m(n, s, t) = n$ if $n < m$ and $\sum_{i=1}^m 1/2^i \leq t$. Therefore, $[f] = e$. \square

Proof of Theorem 2.4 It suffices to show the one direction. Suppose that $f|_n \sim g|_n$ for the restrictions $f|_n, g|_n$ ($n \in \omega$). Then, $(f \cdot g^-)|_n = f|_n \cdot g^-|_n \sim \text{id}_n$ for every $n \in \omega$. By Lemma 2.7, $f \cdot g^- \sim \text{id}_\omega$. Let $\sigma = f(1)$, then $g(1) = \sigma$ and $[g]^{-1} = [(g^-)^{\sigma^{-1}}]$. Hence, $[f][g]^{-1} = [f \cdot (g^-)^{\sigma^{-1}}] = [f \cdot g^-] = [\text{id}_\omega] = e$, i.e. $f \sim g$.

It is known that the abelianization of B_n , i.e. $B_n/(B_n)'$, is isomorphic to \mathbb{Z} . In case of B_ω some difference should occur, since Σ_X coincides with its commutator subgroup $(\Sigma_X)'$ for an infinite X . We don't know whether $B_\omega = (B_\omega)'$ or not. But, the abelianization of B_ω is not isomorphic to \mathbb{Z} , as we shall see in the following.

Some definition is necessary to state the next theorem. An abelian group A is called complete modulo the Ulm subgroup (abbreviated by "complete mod- U "), if for any $x_n \in A$ ($0 < n \in \omega$) with $n! \mid x_{n+1} - x_n$ there exists $x \in A$ such that $n! \mid x - x_n$ for all $0 < n \in \omega$. In other words, $A/U(A)$ is complete [7], where $U(A) = \bigcap_{n \in \omega_+} n!A$. Since any homomorphic image of a complete mod- U abelian group is also complete mod- U , a complete mod- U abelian group has no summand isomorphic to \mathbb{Z} . (See [5] for further information about complete mod- U groups.) This kind of group is related to the first integral singular homology groups of wild spaces [4, 5].

Theorem 2.8. *The abelianization of B_ω is complete modulo the Ulm subgroup.*

It is well-known that $\Sigma_X = (\Sigma_X)'$ for infinite X [10, p. 306]. Therefore, we get the following by Proposition 2.3.

Lemma 2.9. *For an infinite $X \subset \mathbb{C}$, $B_X = P_X(B_X)'$.*

For $f \in A_X$, let $[f]_a$ be the element of the abelianization of B_X corresponding to f , i.e. the map $[f] \rightarrow [f]_a$ is the canonical homomorphism from B_X to $B_X/(B_X)'$. We shall treat with the cases $X = n$ for $n \in \omega$ and $X = \omega$.

Lemma 2.10. *For any pure ω -braid f and neighborhood O of id_ω in F_ω , there exists a pure ω -braid g such that $[g]_a = [f]_a$ and $\text{Im}(g) \subset O$.*

Proof. There exists $n \in \omega$ such that O depends on the i -th co-ordinates ($i \in n$). Reminding the proof of the fact $B_{n+1}/(B_{n+1})' \simeq \mathbb{Z}$, we conclude the existence of a pure $(n+1)$ -braid g' such that $[g']_a = [f|_{n+1}]_a$ and $g'(s)(i) = i$ for $i \in n$ and $s \in \mathbb{I}$. Then, there exists an $(n+1)$ -braid h' such that $[h'] \in (B_{n+1})'$ and $[g'] = [f|_{n+1}][h']$. Since we may assume $|h'(s)(i)| \leq n + 1/2$ for $i \in n$ and $s \in \mathbb{I}$, extend h' to a pure ω -braid h so that $h(s)(i) = i$ for $n + 1 \leq i \in \omega, s \in \mathbb{I}$. Then, $[h] \in (B_\omega)'$ and $(f \cdot h)|_{n+1} \sim g'$. Take H in Definition 1.4 for $(f \cdot h)|_{n+1}$ and g' and define g by: $g(s)(i) = H((f \cdot h)(s)(i), s, 1)$. We have gotten a pure ω -braid g such that $g|_{n+1} = g'$ and $[g] = [f][h]$, which imply $g \in O$ and $[g]_a = [f]_a$. \square

The next lemma is essentially included in [4, Theorem 1.1]. More precisely, what was necessary to prove it is that the one point union of cones $(CX, x) \vee (CY, y)$ satisfies the condition of the next lemma with the common point as base point and that the first integral singular homology group is an abelian group and a homomorphic image of the fundamental group. Therefore, we omit the proof.

Lemma 2.11. *Let Y be a path-connected Hausdorff space and first countable at $y \in Y$. Let A be an abelian group which is a homomorphic image of $\pi_1(Y, y)$ and $h : \pi_1(Y, y) \rightarrow A$ be the homomorphism. Suppose that for any loop f with base point y there exist loops f_n with base point y ($n \in \omega$) such that $h([f_n]) = h([f])$ and $\text{Im}(f_n)$ converge to y . Then, A is complete mod- U .*

Proof of Theorem 2.8. By Lemma 2.9, the abelianization of B_ω is a homomorphic image of P_ω . Now, the theorem follows from Lemmas 2.10 and 2.11.

Remark 2.12. We have not succeeded to prove the torsion-freeness of B_X for uncountable X . As we remarked before Theorem 2.8, we don't know whether $B_\omega = (B_\omega)'$ or not.

3. A REPRESENTATION OF B_ω AS AN AUTOMORPHISM GROUP ON THE UNRESTRICTED FREE PRODUCT

As is well-known, E. Artin [1] represented B_n as an automorphism group on free groups of n -generators. On the other hand, G. Higman [9] introduced a notion "Unrestricted free product". In this section, we represent B_ω as an automorphism group on the unrestricted free product of finitely generated free groups. Let \mathbb{Z}_i ($i \in \omega$) be copies of the integer group \mathbb{Z} and $p_n^m : *_{i \in m} \mathbb{Z}_i \rightarrow *_{i \in n} \mathbb{Z}_i$ the canonical projection for $n \leq m$, where $*_{i \in m} \mathbb{Z}_i$ is the free product of \mathbb{Z}_i 's. We regard $*_{i \in m} \mathbb{Z}_i$ as the trivial

group $\{e\}$ in case $m = 0$ as usual. The unrestricted free product \mathcal{G}_ω is the inverse limit $\lim_{\leftarrow} (*_{i \in n} \mathbb{Z}_i, p_n^m : n \leq m, m, n \in \omega)$. Let $p_E : \mathcal{G}_\omega \rightarrow *_{i \in E} \mathbb{Z}_i$ be the induced projection for a finite subset E of ω . In the following, let δ_i be the generator of \mathbb{Z}_i which corresponds to 1 in \mathbb{Z} . For a group G , we denote the automorphism group of G by $\text{Aut}(G)$.

Theorem 3.1. *Fix an inverse system $(*_{i \in n} \mathbb{Z}_i, p_n^m : n \leq m, m, n \in \omega)$ for \mathcal{G}_ω . Let (\dagger) be a property of $a \in \text{Aut}(\mathcal{G}_\omega)$ as the following:*

- (1) (\dagger) For any $m \in \omega$ there exist a finite subset E of ω , h and σ
- (2) such that $h : *_{i \in m} \mathbb{Z}_i \rightarrow *_{i \in E} \mathbb{Z}_i$ is an isomorphism, $\sigma : m \rightarrow E$
- (3) is a bijection, $p_E a = h p_m$, $h(\delta_i) = U_i^{-1} \delta_{\sigma(i)} U_i$ for some U_i ,
- (4) $h(\prod_{i \in m} \delta_i) = \prod_{i \in E} \delta_i$, where the products are taken under the
- (5) natural ordering on ω .

Then, B_ω is naturally isomorphic to the subgroup of $\text{Aut}(\mathcal{G}_\omega)$ consisting of all automorphisms satisfying (\dagger) .

To show the theorem, some lemmas are necessary.

Lemma 3.2. [10, Theorem 1] *Let $h : \mathcal{G}_\omega \rightarrow \mathcal{F}$ be a homomorphism, where \mathcal{F} is a free group. Then, there exist $m \in \omega$ and a homomorphism $\bar{h} : *_{i \in m} \mathbb{Z}_i \rightarrow \mathcal{F}$ such that $h = \bar{h} p_m$.*

Lemma 3.3. *Suppose that $f_m : \mathbb{I} \rightarrow F_m$ ($m \in \omega_+$) are satisfying $f_m(0) = \text{id}_m$, $\{f_m(1)(i) : i \in m\} \subset \omega$ and $f_m|_n \sim f_n$ for $n \leq m, m, n \in \omega_+$. If $\bigcup_{m \in \omega_+} \{f_m(1)(i) : i \in m\} = \omega$, then there exists $f \in A_\omega$ such that $f|_m \sim f_m$ for every $m \in \omega_+$.*

Proof. By [1, Theorem 6], for each $m \in \omega_+$ there exists $H^m : \mathbb{C} \times \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{C}$ such that

- (1) $H_{s,t}^m : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism, where $H_{s,t}^m(\alpha) = H^m(\alpha, s, t)$;
- (2) $H^m(\alpha, s, 0) = \alpha$ and $H^m(\alpha, 0, t) = H^m(\alpha, 1, t) = \alpha$;
- (3) $H^m(f_{m+1}(s)(i), s, 1) = f_m(s)(i)$ for $i \in m$.

Next, define $f_m^k : \mathbb{I} \rightarrow \mathbb{C}$ as follows: $f_m^0(s) = f_m(s)(m-1)$ and $f_m^k(s) = H^{m-k}(f_m^{k-1}(s), s, 1)$ for $0 < k \in m$. Finally, let $f(s)(m) = f_{m+1}^m(s)$ for $m \in \omega$. It is easy to check that $f(s) \in F_\omega$, $f(1) = \text{id}_\omega$ and $f : \mathbb{I} \rightarrow F_\omega$ is continuous. By definition, $f|_m \sim f_m$ for every $m \in \omega_+$. Now, $\bigcup_{m \in \omega_+} \{f_m(1)(i) : i \in m\} = \omega$ implies $f(1) \in \Sigma_\omega$ and we have shown the lemma. \square

Proof of Theorem 3.1. For $f \in A_\omega$, let $\sigma = f(1)$ and $n \in \omega_+$. The restriction $f|_n$ does not always belong to B_n , but is still a braid in the sense of [1]. Therefore, $f|_n$ induces an isomorphism $e_n^f : *_{i \in n} \mathbb{Z}_i \rightarrow *_{i \in n} \mathbb{Z}_{\sigma(i)}$. For $n \in \omega$, denote the set $\{\sigma(i) : i \in n\}$ by $\sigma[n]$. Then, the following diagram 3.4 commutes, i.e. $e_n^f p_n^m = p_{\sigma[n]}^{\sigma[m]} e_m^f$ for $n \leq m$ ($m, n \in \omega$).

$$\begin{array}{ccc}
 \text{Diagram 3.4} & *_{i \in m} \mathbb{Z}_i & \xrightarrow{e_m^f} *_{i \in \sigma[m]} \mathbb{Z}_i \\
 & \downarrow p_n^m & \downarrow p_{\sigma[n]}^{\sigma[m]} \\
 & *_{i \in n} \mathbb{Z}_i & \xrightarrow{e_n^f} *_{i \in \sigma[n]} \mathbb{Z}_i
 \end{array}$$

Since $\sigma[n] \subset \sigma[m]$ for $n \leq m$ ($m, n \in \omega$) and $\omega = \bigcup_{n \in \omega} \sigma[n]$, $\mathcal{G}_\omega = \lim_{\leftarrow} (*_{i \in \sigma[n]} \mathbb{Z}_i, p_{\sigma[n]}^{\sigma[m]})$: $n \leq m, m, n \in \omega$). Therefore, f induces an automorphism a_f on \mathcal{G}_ω , which satisfies the property (\dagger) by [1, Theorem 15]. It is easy to see that a_f only depends on $[f]$ by Theorem 2.4 and the map $[f] \rightarrow a_f$ is a homomorphism from B_ω to $\text{Aut}(\mathcal{G}_\omega)$. The injectivity of this homomorphism follows again from Theorem 2.4.

To show the surjectivity of this homomorphism, let $a \in \text{Aut}(\mathcal{G}_\omega)$ satisfy the property (\dagger) . Then, there exist h_m and a finite subset E_m of ω which satisfy the properties in (\dagger) for each $m \in \omega$. First, we show $E_n \subset E_m$ for $n \leq m$. For $i \in E_n$, there is $g_i \in \mathcal{G}_\omega$ such that $a(g_i) = \delta_i$, where we identify \mathbb{Z}_i with the corresponding subgroup in \mathcal{G}_ω . Then, $\delta_i = p_{E_n} a(g_i) = h_n p_n(g_i)$ and hence $p_n(g_i) \neq e$, which implies $p_m(g_i) \neq e$. Therefore, $p_{E_m}(\delta_i) = p_{E_m} a(g_i) = h_m p_m(g_i) \neq e$, which implies $i \in E_m$. For $i \in m$, $p_{E_m} a(\delta_i) = h_m p_m(\delta_i) = h_m(\delta_i)$ and hence $p_{E_n}^{E_m} h_m(\delta_i) = p_{E_n}^{E_m} p_{E_m} a(\delta_i) = p_{E_n} a(\delta_i) = h_n p_n(\delta_i) = h_n p_n^m p_m(\delta_i) = h_n p_n^m(\delta_i)$. Therefore, $p_{E_n}^{E_m} h_m = h_n p_n^m$, i.e. the following diagram commutes.

$$\begin{array}{ccc}
 \text{Diagram 3.5} & *_{i \in m} \mathbb{Z}_i & \xrightarrow{h_m} *_{i \in E_m} \mathbb{Z}_i \\
 & \downarrow p_n^m & \downarrow p_{E_n}^{E_m} \\
 & *_{i \in n} \mathbb{Z}_i & \xrightarrow{h_n} *_{i \in E_n} \mathbb{Z}_i
 \end{array}$$

Since h_m satisfies the conditions in (\dagger) and Diagram 3.5, by [1, Theorem 16] there exists a path $f_m : \mathbb{I} \rightarrow F_m$ such that $f_m(0) = \text{id}_m$, $\{f_m(1)(i) : i \in m\} = E_m$, $e_{f_m}^m = h_m$ and $f_m|_n \sim f_n$ for $n \leq m, m, n \in \omega_+$. If we can show that f_m ($m \in \omega_+$) satisfy the conditions of Lemma 3.3, we get $f \in A_\omega$ so that $a = a_f$. Therefore, it suffices to show $\bigcup_{m \in \omega_+} E_m = \omega$. For any $j \in \omega$, there exist $m \in \omega_+$ and $\bar{h} : *_{i \in m} \mathbb{Z}_i \rightarrow \mathbb{Z}_j$ with $p_{\{j\}} a = \bar{h} p_m$ by Lemma 3.2. Take g so that $a(g) = \delta_j$. Then, $\bar{h} p_m(g) = p_{\{j\}} a(g) \neq e$ and hence $p_m(g) \neq e$. Since h_m is an isomorphism, $p_{E_m}(\delta_j) = p_{E_m} a(g) = h_m p_m(g) \neq e$, which implies $j \in E_m$.

Remark 3.4. Instead of \mathcal{G}_ω , there is another candidate to represent B_ω by its automorphism group. It is the fundamental group of the so-called Hawaiian ear ring, which is a subgroup of \mathcal{G}_ω and studied in [5]. But, a natural ω -braid has no naturally corresponding automorphism on the fundamental group of the Hawaiian ear ring. For instance, think of a pure ω -braid such that the first string goes straight, but the others go around the first string.

Supplementary remark In the case of a finite subset X of \mathbb{C} , regarding \mathbb{C}^X as the mapping space from the discrete space X to \mathbb{C} , we can see natural correspondences among $f \in A_X$, a continuous maps $f' : X \times \mathbb{I} \rightarrow \mathbb{C}$ ($f'(x, t) = f(t)(x)$) and a level preserving embedding $f'' : X \times \mathbb{I} \rightarrow \mathbb{C} \times \mathbb{I}$ ($f''(x, t) = (f(t)(x), t)$) whose image $f''(X \times \mathbb{I}) \subset \mathbb{C} \times \mathbb{I}$ coincides with a finite braid in the intuitive sense. For $f, g \in A_X$, a homotopy between f and g in F_X relative to $\{0, 1\}$ corresponds to a level- preserving (ambient, in fact) isotopy between the two embeddings f'' and g'' keeping $X \times \{0, 1\}$ fixed. In case X is infinite, f'' may fail to be an embedding and is just a level preserving continuous injective map in general. And $f \sim g$ if and only if there exists a level preserving homotopy $H_t : X \times \mathbb{I} \rightarrow \mathbb{C} \times \mathbb{I}$ ($t \in \mathbb{I}$) such that $H_0 = f''$, $H_1 = g''$ and H_t is injective. This is our basic view point to infinitary braid groups B_X in this paper.

REFERENCES

- [1] E. Artin, *Theory of braids*, Ann. Math. **48** (1947), 101–126.
- [2] J. S. Birman *Braids, links, and mapping class groups*, Princeton University Press, Princeton, 1974.
- [3] J. L. Dyer *The algebraic braid groups are torsion-free: an algebraic proof*, Math. Zeit. **172**, (1980), 157–160.
- [4] K. Eda *The first integral singular homology groups of one point unions*, Quart. J. Math. **42** (1991), 443–456.
- [5] K. Eda *Free σ -products and non-commutatively slender groups*, J. Algebra **148** (1992), 243–263.
- [6] R. Fox and L. Neuwirth *The braid groups*, Math. Scand. **10** (1962), 119–126.
- [7] L. Fuchs *Infinite abelian groups Vol. 1*, Academic Press, New York, 1970.
- [8] V. L. Hansen *Braids and covering: selected topics*, Cambridge University Press, Cambridge, 1989.
- [9] G. Higman *Unrestricted free products and varieties of topological groups*, J. London Math. Soc. **27**, (1952) 73–81.
- [10] W. R. Scott *Group theory*, Prince-Hall Inc., New Jersey, 1964.
- [11] E. H. Spanier *Algebraic Topology*, McGraw-Hill, New York-San Francisco, 1966.

SCHOOL OF SCIENCE AND ENGINEERING, WASEDA UNIVERSITY, TOKYO 169, JAPAN
E-mail address: eda@waseda.jp

INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA 305, JAPAN